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Oscillation Criteria for Second Order Nonlinear Retarded Differential Equations

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Abstract

The aim of this paper is to deduce oscillatory and asymptotic behavior of the solutions of the second order nonlinear retarded differential equation

$$\Big[r(t)\big|\big[x(t)-p(t)x\left[\tau(t)\right]\big]'\big|^{\alpha-1}\big[x(t)-p(t)x\left[\tau(t)\right]\big]'\Big]'+$$
$$+\,q(t)\big|x\left[\sigma(t)\right]\big|^{\alpha-1}x\left[\sigma(t)\right]=0,$$

where α is a positive constant and $\tau(t)$ and $\sigma(t)$ are delayed arguments.

1 Introduction

In this paper we are concerned with the problem of oscillatory properties of the retarded differential equation of the form

$$\Big[r(t)\big|\big[x(t)-p(t)x\left[\tau(t)\right]\big]'\big|^{\alpha-1}\big[x(t)-p(t)x\left[\tau(t)\right]\big]'\Big]'+$$
$$+\,q(t)\big|x\left[\sigma(t)\right]\big|^{\alpha-1}x\left[\sigma(t)\right]=0. \qquad (E^-)$$

For convenience and further references, we introduce the notation

$$R(t)=\int_{t_0}^{t}\frac{1}{r^{1/\alpha}(s)}\,ds, \quad t\ge t_0.$$

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We suppose throughout the paper that the following hypotheses hold:

$(H1)$ α is a positive constant;

$(H2)$ $\tau(t), \sigma(t) \in C^1\left[t_0, \infty\right)$, $\tau(t) \le t$, $\sigma(t) \le t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$, $\lim\limits_{t\to\infty} \sigma(t) = \infty$, $\sigma'(t) > 0$;

$(H3)$ $r(t) \in C^1[t_0, \infty)$, $r(t) > 0$, $\lim\limits_{t\to\infty} R(t) = \infty$;

$(H4)$ $q(t), p(t) \in C[t_0, \infty)$, $q(t) > 0$, $0 \le p(t) \le p < 1$.

We put $z(t) = x(t) - p(t)x[\tau(t)]$. By a solution of Eq. (E^-) we mean a function $x(t) \in C^1[T_x, \infty)$, $T_x \ge t_0$, which has the property $r(t)\left|z'(t)\right|^{\alpha-1} z'(t) \in C^1[T_x, \infty)$ and satisfies Eq. (E^-) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of Eq. (E^-) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. We assume that (E^-) possesses such a solution.

A solution of (E^-) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise it is said to be nonoscillatory. Eq. (E^-) is said to be oscillatory if every its solution is oscillatory.

This paper is motivated by the papers [4, 7] where the oscillation of differential equations of the form

$$\left[r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right]' + q(t)\left|x\left[\sigma(t)\right]\right|^{\alpha-1}x\left[\sigma(t)\right] = 0 \qquad (E_1)$$

is studied and by the papers [1, 8] where the oscillation criteria for differential equations of the form

$$\left[r(t)\left|\left[x(t) + p(t)x\left[\tau(t)\right]\right]'\right|^{\alpha-1}\left[x(t) + p(t)x\left[\tau(t)\right]\right]'\right]' +$$
$$+ q(t)\left|x\left[\sigma(t)\right]\right|^{\alpha-1}x\left[\sigma(t)\right] = 0, \qquad (E_2)$$

respectively

$$\left[r(t)\left|\left[x(t) + p(t)x\left(t-\tau\right)\right]'\right|^{\alpha-1}\left[x(t) + p(t)x\left(t-\tau\right)\right]'\right]' +$$
$$+ q(t)f\big(x\left[\sigma(t)\right]\big) = 0 \qquad (E_3)$$

with $\dfrac{f(u)}{|u|^{\alpha-1}u} \ge \beta > 0$ for $u \ne 0$, β is a constant, were presented.

2 Main results

We need the following lemma.

Lemma 2.1 (See [5]) *If A and B are nonnegative constants, then*

$$A^{\lambda} - \lambda A B^{\lambda-1} + (\lambda - 1) B^{\lambda} \geq 0, \quad \lambda > 1$$

and the equality holds if and only if $A = B$.

Proof. The case $A = 0$ holds evidently, so we can assume that $A \neq 0$. Then the left side of the inequality can be written in the form

$$1 - \lambda C^{\lambda-1} + (\lambda - 1) C^{\lambda}, \tag{1}$$

where $C = \frac{B}{A}$. Denote (1) by $f(C)$. Clearly (1) is satisfied for $C = 0$. On the other hand, if $C \neq 0$ then function $f(C)$ is decreasing for $C \in (0,1)$ and increasing for $C \in (1, \infty)$. Furthermore $f(1) = 0$. Hence the inequality holds too. The proof is complete. □

The following theorem presents the oscillatory criterion for Eq. (E^{-}).

Theorem 2.1 *Let*

$$\int^{\infty} \left[R^{\alpha} [\sigma(t)]\, q(t) - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{\sigma'(t)}{R[\sigma(t)]\, r^{\frac{1}{\alpha}} [\sigma(t)]} \right] dt = \infty, \tag{2}$$

$$\int^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(u)} \left[\int_{u}^{\infty} q(s)\, ds \right]^{\frac{1}{\alpha}} du = \infty. \tag{3}$$

Then every nonoscillatory solution of Eq. (E^{-}) tends to zero as $t \to \infty$.

Proof. Assume to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (E^{-}). We may assume that $x(t) > 0$. The case of $x(t) < 0$ can be proved by the same arguments.
Set

$$z(t) = x(t) - p(t) x[\tau(t)]. \tag{4}$$

Then $z(t) < x(t)$ and Eq. (E^{-}) can be written in the following form

$$\left[r(t) \left| z'(t) \right|^{\alpha-1} z'(t) \right]' + q(t)\, x^{\alpha} [\sigma(t)] = 0. \tag{5}$$

We claim that $x(t)$ is bounded. To prove it we assume, on the contrary, that $x(t)$ is unbounded. Hence there exists a sequence $\{t_m\}$ such that $\lim_{m\to\infty} t_m = \infty$ moreover $\lim_{m\to\infty} x(t_m) = \infty$ and $x(t_m) = \max\{x(s); t_0 \le s \le t_m\}$. Since $\tau(t) \to \infty$ as $t \to \infty$, we can choose sufficiently large m such that $\tau(t_m) > t_0$. As $\tau(t) \le t$, we have

$$\begin{aligned} x(\tau(t_m)) &\le \max\{x(s);\, t_0 \le s \le \tau(t_m)\} \\ &\le \max\{x(s);\, t_0 \le s \le t_m\} = x(t_m). \end{aligned}$$

Therefore for all large m

$$z(t_m) = x(t_m) - p(t_m)x[\tau(t_m)] \ge (1 - p(t_m))x(t_m).$$

Thus $z(t_m) \to \infty$ as $m \to \infty$.

Eq. (5) implies, that function $r(t)\,|z'(t)|^{\alpha-1}\,z'(t)$ is nonincreasing and we get two possibilities for $z'(t)$:

(i) $z'(t) > 0$,

(ii) $z'(t) < 0$ for $t \ge t_1 \ge t_0$.

The condition (ii) implies that for some positive constant M and $\forall\, t \ge t_1 \ge t_0$

$$r(t)\,|z'(t)|^{\alpha-1}\,z'(t) \le -M < 0.$$

Thus

$$-z'(t) \ge \left(\frac{M}{r(t)}\right)^{\frac{1}{\alpha}}.$$

Integrating the above inequality from t_1 to t, we obtain

$$z(t) \le z(t_1) - M^{\frac{1}{\alpha}}\big(R(t) - R(t_1)\big).$$

Letting $t \to \infty$ in the above inequality and using $(H3)$, we get $z(t) \to -\infty$. This contradiction proves that (i) holds.

For the case (i) we obtain that $z(t) > 0$ and $r(t)\,|z'(t)|^{\alpha-1}\,z'(t) = r(t)\,[z'(t)]^\alpha$. Combining these facts together with $z^\alpha(t) < x^\alpha(t)$, we are led to

$$\big[r(t)\,[z'(t)]^\alpha\big]' + q(t)\,z^\alpha\,[\sigma(t)] \le 0 \tag{6}$$

and

$$\big[r(t)\,[z'(t)]^\alpha\big]' \le 0.$$

Therefore

$$r(t)\left[z'(t)\right]^{\alpha} \le r\left[\sigma(t)\right]\left[z'\left[\sigma(t)\right]\right]^{\alpha},$$

which implies that

$$\frac{z'\left[\sigma(t)\right]}{z'(t)} \ge \left(\frac{r(t)}{r\left[\sigma(t)\right]}\right)^{\frac{1}{\alpha}}. \tag{7}$$

Define

$$w(t) = R^{\alpha}\left[\sigma(t)\right]\frac{r(t)\left[z'(t)\right]^{\alpha}}{z^{\alpha}\left[\sigma(t)\right]} > 0 \tag{8}$$

for $t \ge t_1$.
Differentiating $w(t)$, we have

$$\begin{aligned} w'(t) \quad = \quad & \frac{\alpha R^{\alpha-1}\left[\sigma(t)\right]\sigma'(t)}{r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}\cdot\frac{r(t)\left[z'(t)\right]^{\alpha}}{z^{\alpha}\left[\sigma(t)\right]} + R^{\alpha}\left[\sigma(t)\right]\frac{\left[r(t)[z'(t)]^{\alpha}\right]'}{z^{\alpha}\left[\sigma(t)\right]} \\ & - \alpha R^{\alpha}\left[\sigma(t)\right]\frac{r(t)[z'(t)]^{\alpha}z'\left[\sigma(t)\right]\sigma'(t)}{z^{\alpha+1}\left[\sigma(t)\right]}. \end{aligned} \tag{9}$$

Using (6), (7) and (8), we have

$$\begin{aligned} w'(t) \quad \le \quad & \frac{\alpha\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}w(t) - R^{\alpha}\left[\sigma(t)\right]q(t) \\ & - \frac{\alpha\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}\cdot\frac{R^{\alpha+1}\left[\sigma(t)\right]r^{\frac{\alpha+1}{\alpha}}(t)[z'(t)]^{\alpha+1}}{z^{\alpha+1}\left[\sigma(t)\right]} \\ w'(t) \quad \le \quad & \frac{\alpha\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}\left[w(t) - w^{\frac{\alpha+1}{\alpha}}(t)\right] - \ R^{\alpha}\left[\sigma(t)\right]q(t). \end{aligned} \tag{10}$$

Set $A = w(t)$ and $B = \lambda^{\frac{1}{1-\lambda}}$, where $\lambda = \frac{\alpha+1}{\alpha} > 1$. Applying the Lemma 2.1 to (10), we obtain

$$w'(t) \le \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\cdot\frac{\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]} - R^{\alpha}\left[\sigma(t)\right]q(t).$$

Integrating the above inequality from t_1 to t, we get

$$w(t) \le w(t_1) - \int_{t_1}^{t}\left[R^{\alpha}\left[\sigma(s)\right]q(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\sigma'(s)}{R\left[\sigma(s)\right]r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}\right]ds. \tag{11}$$

Letting $t \to \infty$ in (11), we get $w(t) \to -\infty$ in view of (2). This contradicts to positivity of $w(t)$ and we conclude that $x(t)$ is bounded. Consequently, in view of (4) $z(t)$ is bounded too.

Eq. (5) implies, that function $r(t)\,|z'(t)|^{\alpha-1}\,z'(t)$ is nonincreasing and we get two possibilities for $z'(t)$:

(i) $z'(t) > 0$,

(ii) $z'(t) < 0$ for $t \geq t_2 \geq t_1$.

The condition (ii) implies that for some positive constant N and $\forall\, t \geq t_2$

$$r(t)\,|z'(t)|^{\alpha-1}\,z'(t) \leq -N < 0.$$

Proceeding similarly as in the previous we obtain

$$-z'(t) \geq \left(\frac{N}{r(t)}\right)^{\frac{1}{\alpha}}.$$

Integrating the above inequality from t_2 to t, we obtain

$$z(t) \leq z(t_2) - N^{\frac{1}{\alpha}}\big(R(t) - R(t_2)\big).$$

Letting $t \to \infty$ in the above inequality and using $(H3)$, we get $z(t) \to -\infty$. This contradicts that $z(t)$ is bounded, e.g. (i) holds.

Now we shall discuss the following two cases:

1. $z(t) > 0$,
2. $z(t) < 0$.

Case 1. Let $z(t) > 0$.

Since $z(t)$ is bounded and $z'(t) > 0$, there exists

$$\lim_{t\to\infty} z(t) = 2c, \quad 0 < c < \infty. \tag{12}$$

Integrating (6) from t to ∞ and taking into account monocity of $z^{\alpha}[\sigma(t)]$ and (12) one gets

$$[z'(t)]^{\alpha} \geq c^{\alpha} \cdot \frac{1}{r(t)} \int_t^{\infty} q(s)\,ds.$$

Raising to $\frac{1}{\alpha}$ power and integrating from t_3 to t we acquire

$$z(t) \geq z(t_3) + c \int_{t_3}^{t} \frac{1}{r^{\frac{1}{\alpha}}(u)} \left[\int_u^{\infty} q(s)\,ds\right]^{\frac{1}{\alpha}} du. \tag{13}$$

Letting $t \to \infty$ in the previous inequality, we get $z(t) \to \infty$ in view of (3) and this contradicts the boundedness of the function $z(t)$.

Case 2. Let $z(t) < 0$.

Since $z(t)$ is bounded and $z'(t) > 0$, there exists

$$\lim_{t\to\infty} z(t) = c, \quad -\infty < c \leq 0. \tag{14}$$

The boundedness of $x(t)$ yields $\limsup\limits_{t\to\infty} x(t) = a$, $0 \leq a < \infty$. Then there exists a sequence $\{t_k\}$ such that $\lim\limits_{k\to\infty} t_k = \infty$, $\lim\limits_{k\to\infty} x(t_k) = a$. If $a > 0$, choosing $\epsilon = \frac{a(1-p)}{2p}$ we see that $x[\tau(t)] < a + \epsilon$, eventually. Moreover

$$0 \geq \lim_{k\to\infty} z(t_k) \geq \lim_{k\to\infty} \big(x(t_k) - p(a+\epsilon)\big) = \frac{a}{2}(1-p) > 0.$$

Thus $a = 0$ and that is $\lim\limits_{t\to\infty} x(t) = 0$. $\square$

Now we provide easily verifiable oscillatory criterion for Eq. (E^-).

Corollary 2.1 *Let* (3) *holds and*

$$\liminf_{t\to\infty} \frac{R^{\alpha+1}[\sigma(t)]\, r^{\frac{1}{\alpha}}[\sigma(t)]\, q(t)}{\sigma'(t)} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, \tag{15}$$

Then every nonoscillatory solution of Eq. (E^-) *tends to zero as* $t \to \infty$.

Proof. Let (15) holds. Then there exists $\epsilon > 0$ such that for all large t, say $t \geq t_1$

$$\frac{R^{\alpha+1}[\sigma(t)]\, r^{\frac{1}{\alpha}}[\sigma(t)]\, q(t)}{\sigma'(t)} \geq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \epsilon,$$

which follows that

$$R^{\alpha}[\sigma(t)]\, q(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\sigma'(t)}{R[\sigma(t)]\, r^{\frac{1}{\alpha}}[\sigma(t)]} \geq \epsilon \frac{\sigma'(t)}{R[\sigma(t)]\, r^{\frac{1}{\alpha}}[\sigma(t)]}.$$

Integrating the above inequality from t_1 to t, we obtain

$$\int_{t_1}^{t} \left[R^{\alpha}[\sigma(s)]\, q(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\sigma'(s)}{R[\sigma(s)]\, r^{\frac{1}{\alpha}}[\sigma(s)]}\right] ds \geq$$

$$\geq \epsilon \big[\ln R[\sigma(t)] - \ln R[\sigma(t_1)]\big] \to \infty \quad \text{as} \quad t \to \infty.$$

Now the assertion of Corollary 2.1 follows from Theorem 2.1. $\square$

Corollary 2.2 *If*

$$\int^{\infty}\left[[\sigma(s)]^{\alpha}q(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\sigma'(s)}{\sigma(s)}\right]ds=\infty, \tag{16}$$

$$\int^{\infty}\left[\int_{u}^{\infty}q(s)\,ds\right]^{\frac{1}{\alpha}}du=\infty, \tag{17}$$

then every nonoscillatory solution of Eq. (18)

$$\left[\left|\left[x(t)-p(t)x\left[\tau(t)\right]\right]'\right|^{\alpha-1}\left[x(t)-p(t)x\left[\tau(t)\right]\right]'\right]'+$$

$$+\,q(t)\big|x\left[\sigma(t)\right]\big|^{\alpha-1}x\left[\sigma(t)\right]=0 \tag{18}$$

tends to zero as $t\to\infty$.

Proof. It is easy to see that the conditions (2) and (3) reduce to (16) and (17) for $r(t)\equiv 1$. □

Corollary 2.3 *If*

$$\int^{\infty}\left[R\left[\sigma(s)\right]q(s)-\frac{\sigma'(s)}{4R\left[\sigma(s)\right]r\left[\sigma(s)\right]}\right]ds=\infty, \tag{19}$$

$$\int^{\infty}\frac{1}{r(u)}\int_{u}^{\infty}q(s)\,ds\,du=\infty, \tag{20}$$

then every nonoscillatory solution of Eq. (21)

$$\left[r(t)\left[x(t)-p(t)x\left[\tau(t)\right]\right]'\right]'+q(t)x\left[\sigma(t)\right]=0 \tag{21}$$

tends to zero as $t\to\infty$.

Proof. It is easy to see that (2) and (3) reduce to (19) and (20) for $\alpha=1$. □

Corollary 2.4 *Let* (2) *and* (3) *hold. If* $p(t)$ *oscilates, then Eq.* (E^{-}) *is oscillatory.*

Proof. Let $x(t)$ is a positive solution of (E^-). Arguing exactly as in the proof of Theorem 2.1 we can show that $z(t) < 0$. If $\{t_k\}$ is a sequence of zeros of $p(t)$, then

$$0 > z(t_k) = x(t_k) - p(t_k)x(\tau(t_k)) > 0.$$

That is a contradiction. □

Now we will use so-called the integral averaging technique. Let us consider a function $H(t,s)$ satisfying the following properties

(i) $H(t,s) > 0$ for $t > s \geq t_0$,

(ii) $H(t,t) = 0$ and $\frac{\partial H(t,s)}{\partial s} < 0$.

Denote

$$\begin{aligned} h(t,s) &= \frac{-\frac{\partial H(t,s)}{\partial s}}{\sqrt{H(t,s)}}, \\ Q(t,s) &= \sqrt{H(t,s)} \cdot \frac{\alpha\sigma'(s)}{R[\sigma(s)]r^{\frac{1}{\alpha}}[\sigma(s)]} - h(t,s), \quad \text{for } t > s. \end{aligned}$$

Theorem 2.2 *Let $\alpha \geq 1$ and (3) holds. Assume that for some $k \in (0,1)$*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)R^{\alpha}[\sigma(s)]\,q(s) - \frac{R[\sigma(s)]\,r^{\frac{1}{\alpha}}[\sigma(s)]}{4\alpha k\sigma'(s)} Q^2(t,s) \right] ds = \infty. \tag{22}$$

Then every nonoscillatory solution of Eq. (E^-) tends to zero as $t \to \infty$.

Proof. Assume to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (E^-). Without loss of generality we may assume that $x(t) > 0$. Proceeding similarly as in the proof of Theorem 2.1 we have $z(t) > 0$, $z'(t) > 0$ and using the fact that $[r(t)\,(z'(t))^{\alpha}]^{\frac{1}{\alpha}}$ is nonincreasing, we see that for any $k_1 \in (0,1)$ and for all large t $(t \geq t_1)$

$$\begin{aligned} z[\sigma(t)] &\geq \int_{t_1}^{\sigma(t)} z'(s)ds = \int_{t_1}^{\sigma(t)} \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(r^{\frac{1}{\alpha}}(s)z'(s) \right) ds \\ &\geq r^{\frac{1}{\alpha}}[\sigma(t)]\,z'[\sigma(t)]\,\big(R[\sigma(t)] - R(t_1)\big) \\ &> k_1 R[\sigma(t)]\,r^{\frac{1}{\alpha}}[\sigma(t)]\,z'[\sigma(t)]. \end{aligned} \tag{23}$$

Taking into account (23) and the monotonicity of $r(t)\left[z'(t)\right]^{\alpha}$, we conclude that

$$\begin{aligned}\frac{z'\left[\sigma(t)\right]}{z\left[\sigma(t)\right]} &= \frac{1}{r\left[\sigma(t)\right]}\cdot\frac{r\left[\sigma(t)\right]\left[z'\left[\sigma(t)\right]\right]^{\alpha}}{z^{\alpha}\left[\sigma(t)\right]}\cdot\left(\frac{z\left[\sigma(t)\right]}{z'\left[\sigma(t)\right]}\right)^{\alpha-1} \\ &\geq \frac{r(t)\left[z'(t)\right]^{\alpha}}{z^{\alpha}\left[\sigma(t)\right]}\cdot\frac{kR^{\alpha-1}\left[\sigma(t)\right]}{r^{\frac{1}{\alpha}}\left[\sigma(t)\right]} \geq \frac{k}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}w(t) \qquad (24)\end{aligned}$$

where $k = k_1^{\alpha-1} \in (0,1)$.

Using the function $w(t)$ defined in (8), $w'(t)$ in (9) and the inequality (24) we obtain

$$\begin{aligned}w'(t) &\leq \frac{\alpha\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}w(t) - R^{\alpha}\left[\sigma(t)\right]q(t) \\ &\quad - \alpha R^{\alpha}\left[\sigma(t)\right]\sigma'(t)\frac{r(t)\left[z'(t)\right]^{\alpha}}{z^{\alpha}\left[\sigma(t)\right]}\cdot\frac{z'\left[\sigma(t)\right]}{z\left[\sigma(t)\right]} \\ &\leq \frac{\alpha\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}w(t) - R^{\alpha}\left[\sigma(t)\right]q(t) - \frac{\alpha k\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}w^2(t).\end{aligned}$$

Multiplying this inequality with $H(t,s) > 0$ and following integrating from t_1 to t we have

$$\begin{aligned}&\int_{t_1}^{t} H(t,s)R^{\alpha}\left[\sigma(s)\right]q(s)ds \leq \\ &\leq \int_{t_1}^{t} H(t,s)\frac{\alpha\sigma'(s)}{R\left[\sigma(s)\right]r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}w(s)ds \\ &\quad - \int_{t_1}^{t} H(t,s)\frac{\alpha k\sigma'(s)}{R\left[\sigma(s)\right]r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}w^2(s)ds - \int_{t_1}^{t} H(t,s)w'(s)ds.\end{aligned}$$

Now integrating (per partes) from t_1 to t and using definition of the functions $h(t,s)$ and $Q(t,s)$ we are led to

$$\begin{aligned}&\int_{t_1}^{t} H(t,s)R^{\alpha}\left[\sigma(s)\right]q(s)ds \leq \\ &\leq H(t,t_1)w(t_1) - \int_{t_1}^{t} H(t,s)\frac{\alpha k\sigma'(s)}{R\left[\sigma(s)\right]r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}w^2(s)ds \\ &\quad + \int_{t_1}^{t}\sqrt{H(t,s)}\left[\sqrt{H(t,s)}\cdot\frac{\alpha\sigma'(s)}{R\left[\sigma(s)\right]r^{\frac{1}{\alpha}}\left[\sigma(s)\right]} - h(t,s)\right]w(s)ds \leq\end{aligned}$$

$$\leq H(t,t_1)w(t_1) - \int_{t_1}^{t} H(t,s)\frac{\alpha k\sigma'(s)}{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}w^2(s)ds + \int_{t_1}^{t}\sqrt{H(t,s)}Q(t,s)w(s)ds.$$

Consequently

$$\int_{t_1}^{t} H(t,s)R^{\alpha}\left[\sigma(s)\right]q(s)ds \leq$$

$$\leq H(t,t_1)w(t_1) + \int_{t_1}^{t}\frac{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}{4\alpha k\sigma'(s)}Q^2(t,s)ds$$

$$-\int_{t_1}^{t}\left[\sqrt{H(t,s)\frac{\alpha k\sigma'(s)}{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}}w(s) - \frac{1}{2}\sqrt{\frac{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}{\alpha k\sigma'(s)}}Q(t,s)\right]^2 ds.$$

Therefore

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)R^{\alpha}\left[\sigma(s)\right]q(s) - \frac{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}{4\alpha k\sigma'(s)}Q^2(t,s)\right]ds \leq w(t_1).$$

Letting $t\to\infty$ we get the contradiction with (22). The rest of proof is similar to the proof of Theorem 2.2. □

Let us have $H(t,s)$ defined by (25).

$$H(t,s) = (t-s)^n, \qquad n \text{ is a positive integer.} \tag{25}$$

Then Theorem 2.2 provides the following criterion:

Theorem 2.3 *Let $\alpha \geq 1$ and (3) holds. Assume that for some $k \in (0,1)$*

$$\limsup_{t\to\infty}\frac{1}{(t-t_1)^n}\int_{t_1}^{t}\left[(t-s)^n R^{\alpha}\left[\sigma(s)\right]q(s) - \frac{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}{4\alpha k\sigma'(s)}Q^2(t,s)\right]ds = \infty, \tag{26}$$

where

$$Q(t,s) = (t-s)^{\frac{n}{2}} \left(\frac{\alpha\sigma'(s)}{R\big[\sigma(s)\big] r^{\frac{1}{\alpha}}[\sigma(s)]} - \frac{n}{t-s} \right).$$

Then every nonoscillatory solution of Eq. (E^-) *tends to zero as* $t \to \infty$.

Remark 1 *Theorem 2.1 extends results presented for neutral differential equations of the forms*

$$\big(x(t) - px(t-\tau)\big)'' + q(t)x\left[\sigma(t)\right] = 0,$$

$$\big(x(t) \pm p(t)x\left[\tau(t)\right]\big)^{(n)} + q(t)x\left[\sigma(t)\right] = 0,$$

$$\big(x(t) - p(t)x\left[\tau(t)\right]\big)^{(n)} + q(t)f\big(x\left[\sigma(t)\right]\big) = 0$$

presented in [3], [2] *and* [6].

Remark 2 *Putting* $p(t) \equiv 0$, *Theorem 2.1 generalizes results presented in* [4] *and* [7], *where the differential equations of the form* (E_1) *are studied.*

Remark 3 *Theorems 2.1, 2.2 and 2.3 complement results presented in* [1, 8], *where authors deal with the neutral differential equations of the form* (E_2), *respectively* (E_3).

Example 1 *We consider differential equation*

$$\left[\left|\left(x(t) - px\left(\frac{t}{2}\right)\right)'\right|^{\alpha-1} \left(x(t) - px\left(\frac{t}{2}\right)\right)'\right]' +$$

$$+\frac{2\alpha\beta^{\alpha}(2p-1)^{\alpha}}{t^{\alpha+1}} \big|x(\beta t)\big|^{\alpha-1} x(\beta t) = 0, \tag{27}$$

with $t > 0$, $r(t) = 1$, $\tau(t) = \frac{t}{2}$, $p(t) = p$, $\frac{1}{2} < p < 1$, $\sigma(t) = \beta t$, $0 < \beta < 1$, $q(t) = \frac{2\alpha\beta^{\alpha}(2p-1)^{\alpha}}{t^{\alpha+1}}$. *If*

$$2\alpha\beta^{2\alpha}(2p-1)^{\alpha} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

then by Theorem 2.1 every nonoscillatory solution of Eq. (27) *tends to zero as* $t \to \infty$. *One of the solutions of Eq.* (27) *is for example* $x(t) = \frac{1}{t}$.

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